# Exam 1 key

1.) consider the dependence equation $c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}4\\5\\6\end{pmatrix}+c_3\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow \left(\begin{array}{ccc|c}1&4&2&0\\2&5&1&0\\3&6&0&0\end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c}1&0&-2&0\\0&1&1&0\\0&0&0&0\end{array}\right) \Rightarrow \begin{array}{l}c_1 = 2c_3\\ c_2 = -c_3\\ c_3 = c_3 \leftarrow \text{free variable}\end{array}$$

so linear dependent

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2.) like in (1) consider dependence eq. $c_1\begin{pmatrix}5\\0\\0\end{pmatrix}+c_2\begin{pmatrix}7\\2\\-6\end{pmatrix}+c_3\begin{pmatrix}9\\4\\-8\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow \left(\begin{array}{ccc|c}5&7&9&0\\0&2&4&0\\0&-6&-8&0\end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right) \Rightarrow c_1=c_2=c_3=0$$

so linear independent

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3.) let $f, g \in C^4(\mathbb{R})$, i.e. $4^{th}$ differentiable fncs.

then $T(f+g) = \frac{d^4}{dx^4}(f+g) = \frac{d^4 f}{dx^4} + \frac{d^4 g}{dx^4} = T(f) + T(g)$

let $c$ be a scalar so $T(cf) = \frac{d^4}{dx^4}(cf) = c\frac{d^4 f}{dx^4} = cT(f)$

thus $T$ is linear.

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4.) let $f, g$ be contin. fncs.

then $T(f+g) = \sum_{j=1}^{n}(f+g)(x_j) = \sum_{j=1}^{n}[f(x_j)+g(x_j)] = \sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} g(x_j)$

$= T(f) + T(g)$

let $c$ be a scalar, so $T(cf) = \sum_{j=1}^{n}(cf)(x_j) = \sum_{j=1}^{n} c f(x_j) = c\sum_{j=1}^{n} f(x_j) = cT(f)$

so $T$ is linear.

5.) if $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ then we compute $T(e_1)$, $T(e_2)$, $T(e_3)$, $T(e_4)$

w/ $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$T(e_1) = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} -1 \\ 10 \\ 4 \\ 11 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $T(e_4) = \begin{pmatrix} 0 \\ 0 \\ 5 \\ -8 \end{pmatrix}$

so $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 10 & 2 & 0 \\ 0 & 4 & 0 & 5 \\ 0 & 11 & 0 & -8 \end{pmatrix}$

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6.) if $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, like (5) we compute $T(e_1)$, $T(e_2)$, $T(e_3)$

here $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

so $A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 3 & -2 \end{pmatrix}$

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(7.) set up $\{A \mid I_3\} = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 4R_2} \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right)$

$\xrightarrow{R_3 + 3R_1} \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right)$

$\xrightarrow[R_2 - 3R_3]{R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -2 & 4 & -1 \\ 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right) \xrightarrow{\text{swap rows}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right)$

so $A^{-1} = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$

8.) ~~set~~ set up $[A \mid I_3] = \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow[R_2+3R_1]{R_3-2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_3+3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 3 & 1 \end{array}\right)$$

$$\xrightarrow[R_2+R_3]{R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & 1 \\ 0 & 1 & 0 & 7 & 4 & 1 \\ 0 & 0 & 2 & 4 & 3 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & 1 \\ 0 & 1 & 0 & 7 & 4 & 1 \\ 0 & 0 & 1 & 2 & \frac{3}{2} & \frac{1}{2} \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} 5 & 3 & 1 \\ 7 & 4 & 1 \\ 2 & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

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9.) We know that $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \Rightarrow c_1 = c_2 = \cdots = c_k = 0$ since $v_1, \ldots, v_k$ lin. indep.

so apply $A$ to both sides to get $A(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = A(0)$

but $A$ is a lin. trans. $\Rightarrow c_1(Av_1) + c_2(Av_2) + \cdots + c_k(Av_k) = 0$

but $c_1 = \cdots = c_k = 0$ from above, thus, $Av_1, \ldots, Av_k$ are lin. indep.